Math 245C Lecture 25 Notes

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1 Extensions and Transformations of Distributions

1.1 Extension of distributions

Let $U \subseteq \mathbb{R}^n$ be open. If $V \subseteq U$ is open and $T, S \in \mathcal{D}'(U)$, we say that $T = S$ on V if $T|_{C_c^\infty(V)} = S|_{C_c^\infty(V)}$. Assume $V_1, V_2 \subseteq U$ are open and $T, S \in \mathcal{D}'(U)$ are such that $T|_{C_c^\infty(V_1)} = S|_{C_c^\infty(V_1)}$ and $T|_{C_c^\infty(V_2)} = S|_{C_c^\infty(V_2)}$.We want to show that $T|_{C_c^\infty(V_1 \cup V_2)} = S|_{C_c^\infty(V_1 \cup V_2)}$.

Here is a wrong proof: Let $\phi \in C_c^\infty(V)$, and assume that $V_1 \cap V_2 = \varnothing$. Then

$$T(\varphi) = T(\mathbb{1}_{V_1}\phi + \mathbb{1}_{V_2}\phi) = T(\mathbb{1}_{V_1}\phi) + T(\mathbb{1}_{V_2}\phi) = S(\mathbb{1}_{V_1}\phi) + S(\mathbb{1}_{V_2}\phi) = S(\varphi).$$

This is not a correct proof because $\mathbb{1}_{V_1}\phi$ need not be in C_c^∞.

Theorem 1.1. *Let $(V_\alpha)_{\alpha \in I}$ be open subsets of U and let $V = \bigcup_{\alpha \in I} V_\alpha$. Let $T, S \in \mathcal{D}'(V)$ be such that $T|_{C_c^\infty(V_\alpha)} = S|_{C_c^\infty(V_\alpha)}$ for all $\alpha \in I$. Then $T|_{C_c^\infty(V)} = S|_{C_c^\infty(V)}$.*

Proof. Let $\phi \in C_c^\infty(V)$. We are to show that $T(\phi) = S(\phi)$. Set $K = \operatorname{supp}(\phi) \subseteq V = \bigcup_{\alpha \in I} V_\alpha$. Since K is compact, there are $\alpha_1, \dots, \alpha_m \in I$ such that $K \subseteq \bigcup_{j=1}^m V_{\alpha_j}$. For each $x \in K$, there exist $r(x) > 0$ and $j \in \{1, \dots, m\}$ such that $B_{2r(x)}(x) \subseteq V_{\alpha_j}$. Note that $K \subseteq \bigcup_{x \in K} B_{r(x)}(x)$, and so there exists $x_1, \dots, x_\ell \in K$ such that $K \subseteq \bigcup_{i=1}^\ell B_{r(x_i)}(x_i)$.

For each $j \in \{1, \dots, m\}$, set $I_j = \{i \in \{1, \dots, \ell\} : B_{2r(x_i)}(x_i) \subseteq V_{\alpha_j}$. Note that the set $K_j := \bigcup_{i \in I_j} \overline{B_{r(x_i)}(x_i)}$ is compact, and $K_j \subseteq V_{\alpha_j}$. By the extended Urysohn's lemma, there exists $f_j \in C_c^\infty(\mathbb{R}^n, [0,1])$ such that $f_j|_{K_j} \equiv 1$ and $\operatorname{supp}(f_j) \subseteq V_{\alpha_j}$. Set $\mathcal{E} = \{\sum_{j=1}^m f_j > 0\}$. On K, $\sum_{j=1}^m f_j \geq 1$, and so $K \subseteq \mathcal{E}$. We apply the extended Urysohn's lemma once more to obtain $f \in C_c^\infty(\mathbb{R}^n)$ such that $f|_K \equiv 1$ and $\operatorname{supp}(f) \subseteq \mathcal{E}$. Set $f_{m+1} = 1 - f$. Now $f_1 + \cdots + f_{m+1}$ is always strictly positive because $f_1 + \cdots + f_m > 0$ on $\mathcal{E}$ and 1 outside $\mathcal{E}$.

We can hence define

$$h_j = \frac{f_j}{\sum_{i=1}^{m+1} f_i} \in C_c^\infty(\mathbb{R}^n, [0,1])$$

for each $1 \leq j \leq m$. Note that $\mathrm{supp}(h_j) \subseteq V_{\alpha_j}$ and that $(\sum_{j=1}^m h_j)|_K \equiv 1$. Thus, $\phi = \phi \sum_{j=1}^m h_j$, so

$$T(\phi) = T\left(\phi \sum_{j=1}^m h_j\right) = \sum_{j=1}^m T(\phi h_j) = \sum_{j=1}^m S(\phi h_j) = S\left(\phi \sum_{j=1}^m h_j\right) = S(\phi). \qquad \square$$

1.2 Transformations of distirbutions

Definition 1.1. Let $T \in \mathcal{D}'(U)$. If $\alpha \in \mathbb{N}^n$ is a multi-index, define $\partial^\alpha T : C_c^\infty(U) \to \mathbb{R}$ as

$$(\partial^\alpha)T(\phi) = (-1)^{|\alpha|} T(\partial^\alpha \phi).$$

Definition 1.2. If $\psi \in C^\infty(U)$ and $T \in \mathcal{D}'(U)$, define $\psi T : C_c^\infty(U) \to \mathbb{R}$ as

$$(\psi T)(\phi) = T(\psi\phi), \qquad \phi \in C_c^\infty(U).$$

Definition 1.3. If $y \in \mathbb{R}^n$, we define $\tau_y(T) : C_c^\infty(U - y) \to \mathbb{R}$ as

$$\tau_y(T)(\phi) = T(\tau_{-y}\phi), \qquad \phi \in C_c^\infty(U).$$

Definition 1.4. Let $S : \mathbb{R}^N \to \mathbb{R}^n$ be a linear bijection, and set $V = S^{-1}(U)$. We define $T \circ S : C_c^\infty(V) \to \mathbb{R}$ as

$$T \circ S(\phi) = \frac{1}{|\det(S)|} T(\phi \circ S^{-1}), \qquad \phi \in C_c^\infty(V).$$

Theorem 1.2. *Let T, S, ψ, y, α be as above. Then*

1. $\partial^\alpha T, \psi T \in \mathcal{D}'(U)$
2. $\tau_y(T) \in \mathcal{D}'(U - y)$
3. $T \circ S \in \mathcal{D}'(V)$.

Proof. For the second statement, the idea is that τ_y is an isometry of $C_c^\infty(U) \to C_c^\infty(U - y)$.

For the third statement, the idea is that $|\det(S)|^{-1}\phi \circ S^{-1}$ is an isomorphism of $C_c^\infty(V)$ into $C_c^\infty(U)$.

For the first statement, let's take something weaker, say $g \in L^p(U)$. Then $\partial_{x_i} g$ exists as a distribution. Can we represent $\partial_{x_i g}$ as an L^p function? If we can, say $g \in W^{1,p}(U)$. Similarly, if $\partial^2_{x_I, x_j} g \in L^p(U)$, then say that $g \in W^{2,p}$. $\square$

We will continue this next time. This will involve **Sobolev spaces**.